INVESTIGATION OF FREE OSCILLATIONS IN AN AUTONOMOUS SYSTEM OF COORDINATES FOR A MOVING OBJECT

(ISSLEDOVANIE SVOBODNYKH KOLEBANII V SISTEME AVTONOMNOGO OPREDELENIIA KOORDINAT DVIZHUSHCHEGOSIA OB''EKTA)

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Considered is the effect of gyrocompass free oscillations upon a system as a whole, with consideration of a possible error in the initial conditions of the integrating device. This problem, formulated by Ishlinskii [1], is solved for an orbitrary maneuver of the base. A simple relation between the calculated trajectory and the motion of the compass vertical is obtained.

The system, suggested in [1] is considered. Section 1 describes system aspects most applicable to the investigations being carried out. Sections 2 and 3 prove two statements which reduce the problem from system oscillations as a whole to oscillations of the gyrocompass vertical. The mechanical model for the compass is given in Section 4.

The investigation is carried out assuming all elements of the system to be ideal.

1. The problem of determining earth coordinates for an object is equivalent to the determination of its location on a stationary sphere S, coinciding with the earth surface, but not taking part in the diurnal rotation of the earth. If the coordinate network for the sphere S coincides with the earth geographical coordinate network at $t = 0$, then the object with the latitude $\phi(t)$ and the longitude $\lambda(t)$ on the network of sphere S has the current earth coordinates $\phi(t)$, $\lambda(t) - Ut$, respectively, where U is the angular velocity of earth diurnal rotation. This transfer causes no error, therefore in the following we consider the problem of determining the location on the sphere S.

For the solution of this problem, [1] suggests the use of a digital computer, a horizon compass and a directional gyroscope.

The digital computer solves the system of equations

$$\frac{d\varphi}{dt} = \omega_y(t)\sin\vartheta, \qquad \frac{d\vartheta}{dt} = \omega_z(t) - \omega_y(t)\cos\vartheta\tan\varphi, \qquad \frac{d\lambda}{dt} = \omega_y(t)\frac{\cos\vartheta}{\cos\varphi} \tag{1.1}$$

If ϕ_0, λ_0, ϑ_0 are given as initial conditions for ϕ, λ, ϑ (Fig. 1), and if the coefficients are taken as certain functions $\omega_y(t)$ and $\omega_z(t)$, then the computer output yields the current coordinates $\phi(t)$, $\lambda(t)$ of a certain spherical trajectory. This computed trajectory originates from the point ϕ_0, λ_0 at an angle ϑ_0 towards the parallel, and in this motion the natural Darboux trihedron would have

$$\omega_y = \omega_y(t), \qquad \omega_z = \omega_z(t)$$

Such a trajectory is unique.

The horizon compass and the directional compass give the angular velocities of the compass trihedron ω_y and ω_z. The quantity ω_y is related to the angle ϵ for separation of the gyroscopes by

$$\omega_y = \frac{2B\cos\varepsilon}{amR}$$

and ω_z is measured directly by the directional gyroscope, the sensitive axis of which is rigidly attached to the compass trihedron. The ω_y- and ω_z-magnitudes are supplied to the computer as coefficients.

Note that here and in the following the symbols $\omega_y[\omega_z]$ denote the angular velocity of the trihedron about its own axis $y[z]$; it is always specified, however, which trihedron is meant.

If definite initial conditions are satisfied for the compass [2], then the compass trihedron coincides with the natural Darboux trihedron for any motion of the base on the sphere S, and the computer is given ω_y and ω_z for the Darboux trihedron of the base trajectory. If, at the same time, the given initial conditions ϕ_0, λ_0, ϑ_0 correspond to the initial position of the base and to the initial direction of its velocity, then, according to the stated characteristics of the computer, the output of the computer will yield the current coordinates for the true trajectory of the base.

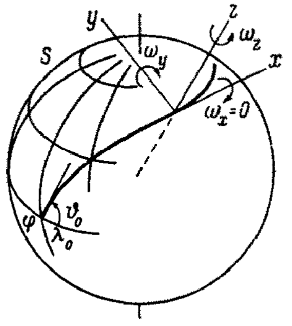

Fig. 1.

If some initial conditions are not satisfied, the computed trajectory of the compass or the computer does not coincide with the true trajectory, but is related to it in a certain way. The character of this relationship is clarified in the following sections.

2. Let us consider the case when the required initial conditions for the compass are satisfied and the computer is given ω_y and ω_z for the Darboux trihedron of the true base trajectory. We will show that in this case for arbitrary introduction of ϕ_0, λ_0, ϑ_0 the computed trajectory is congruent with the true trajectory.

In accordance with the characteristics of the computer, in this case the $\omega_y(t)$ and $\omega_z(t)$ of the Darboux trihedron for the computed trajectory are identically equal to the corresponding angular velocities of the Darboux trihedron for the true trajectory. It follows, therefore, that, independently of ϕ_0, λ_0, ϑ_0, the following are identically equal in pairs for both trajectories:

a) Velocities $V(t)$, since $V(t) = R\,\omega_y(t)$;

b) Traversed distances $s(t)$, since $s(t)$ is a definite integral of $V(t)$;

c) Geodesic curvatures as functions of time $K_g(t)$, since

$$K_g(t) = \omega_z(t) / V(t)$$

d) Geodesic curvatures as functions of distances $K_g(s)$, since $K_g(s)$ is already given parametrically by functions $K(t)$ and $s(t)$.

The last point indicates congruency.

From the proof presented it follows that the general solution of system (1.1) for an arbitrary fixed pair of functions $\omega_y(t)$, $\omega_z(t)$ is obtained from any particular solution by recording it on an arbitrarily rotated spherical system of coordinates. Three constants, characterizing this rotation, play the role of three arbitrary constants in the general solution.

Two close particular solutions (Fig. 2) are obtained from each other by rotation through a small angle δ, depending only upon the difference in the initial conditions of ϕ, λ, ϑ, and the angular distance along the great circle between any points of these solutions corresponding to equal times does not exceed Δ. Obviously $\Delta = \delta$. Thus, the question to be considered is the stability of any particular solution.

It is important for the problem considered that, as was stated before, the existence of an error in the initial conditions ϕ_0, λ_0, ϑ_0 does not produce a cumulative error in the determination of current location.

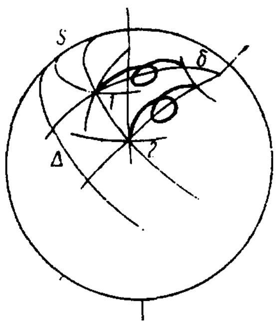

Fig. 2.

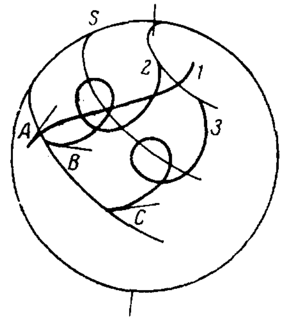

Fig. 3.

Maximum possible error does not exceed some angle δ dependent only on the initial error. The smaller the initial error, the smaller is the angle δ and it is zero when there is no initial error.

3. We will prove the following statement for the general case when the necessary initial conditions for the compass are likewise unsatisfied, in order to utilize the results from the previous section.

The trihedron of a compass oscillating because of unsatisfied initial conditions, realizes physically the Darboux trihedron for the spherical trajectory of that point on the sphere S, whose true location vertical is at each moment colinear with that of the compass. In short, the axes of the compass trihedron are at each moment parallel to the corresponding axes of the Darboux trihedron for the indicated trajectory.

If $\mathbf{r}_1(t)$ is a unit vector of the z-axis for the Darboux trihedron, then its x-axis is directed along $\dot{\mathbf{r}}_1(t)$ and y-axis along $\mathbf{r}_1(t) \times \mathbf{r}_1(t)$. In accordance with the compass equations [2-5], $\omega_x = 0$ for the compass trihedron, therefore if $\mathbf{r}_2(t)$ is a unit vector for the z-axis of the compass trihedron, then its x-axis is directed along $\dot{\mathbf{r}}_2(t)$ and the y-axis along $\mathbf{r}_2(t) \times \dot{\mathbf{r}}_2(t)$.

For the trihedrons considered $\mathbf{r}_1(t) \equiv \mathbf{r}_2(t)$, whence

$$\dot{\mathbf{r}}_1(t) \equiv \dot{\mathbf{r}}_2(t), \quad \mathbf{r}_1(t) \times \dot{\mathbf{r}}_1(t) \equiv \mathbf{r}_2(t) \times \mathbf{r}_2(t)$$

This proves the colinearity of the corresponding axes.

It follows directly from the proof that, in the general case, the computer receives $\omega_y(t)$ and $\omega_z(t)$ for the Darboux trajectory given in the formulation of the statement, and in accordance with Section 2 its congruent trajectory is calculable.

Thus, in the general case, when there is an error in the initial conditions of the compass and in the initial values ϕ, λ, ϑ, one can say the following about system performance (Fig. 3). A compass whose base is moving along an arbitrary trajectory A and which is oscillating because of unsatisfied initial conditions yields the parameters ($\omega_y(t)$ and $\omega_z(t)$ for the Darboux trihedron) for a different trajectory B. This trajectory on the sphere S would be described by a vector of the compass vertical originating at the center of the sphere. The digital computer yields the current coordinates for the trajectory C, which is the same as B except that it is rotated on the sphere S in accordance with the given initial conditions for ϕ, λ, ϑ. These initial conditions indicate the origin and the direction of the trajectory. In particular, one may say that for any maneuvering of the base the considered system of autonomous coordinates has no other cumulative error than that which the compass may have in defining the location vertical.

4. In conclusion, we introduce the mechanical analogy for an ideal horizon gyrocompass. Such an analogy permits a more descriptive interpretation of the compass vertical behavior for arbitrary base maneuvers and for proof of the stability of its coincidence with the location vertical for the case of simple courses [missions].

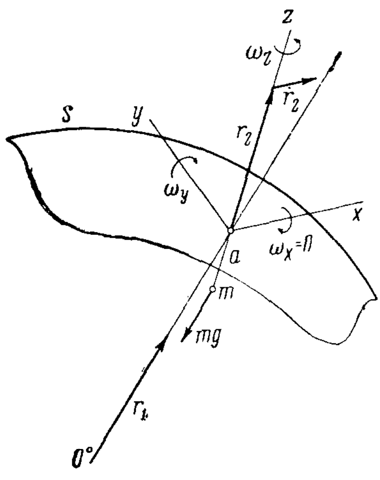

Fig. 4.

An arbitrary maneuver of the base on sphere S may be prescribed by the unit vector of the local vertical $\mathbf{r}_1(t)$ passed from the center 0° of sphere S (Fig. 4). The behavior of an ideal horizon gyrocompass in this maneuver is fully characterized by the unit vector of the compass vertical. Indeed, if $\mathbf{r}_2(t)$ is the unit vector for the z-axis of the compass trihedron, then on the strength of the identity $\omega_x = 0$ we obtain for the unit vectors of the x- and y-axes the expressions

$$[x] = \frac{\dot{\mathbf{r}}_2}{|\dot{\mathbf{r}}_2|}, \qquad [y] = \frac{\mathbf{r}_2 \times \dot{\mathbf{r}}_2}{|\dot{\mathbf{r}}_2|} \tag{4.1}$$

and the angular velocities of the compass trihedron are

$$\omega_y = |\dot{\mathbf{r}}_2|, \qquad \omega_z = \frac{\ddot{\mathbf{r}}_2 (\mathbf{r}_2 \times \dot{\mathbf{r}}_2)}{\dot{\mathbf{r}}_2^2} \tag{4.2}$$

Thus, all quantities entering into the compass equations

$$\frac{d}{dt}(2B\cos\varepsilon) = M_y, \qquad \omega_x = 0$$
$$\omega_z 2B\cos\varepsilon = -M_x, \qquad \omega_y = \frac{2B\cos}{amR} \tag{4.3}$$

may be expressed by means of $\mathbf{r}_1(t)$, $\mathbf{r}_2(t)$ and their derivatives.

It should be remembered that M_x and M_y are moments created by the gravity forces $mg\mathbf{r}_1$ and inertia forces in $mR\mathbf{r}_1$, i.e.

$$\mathbf{F} = -mR\left(\ddot{\mathbf{r}}_1 + \frac{g}{R}\mathbf{r}_1\right)$$

and the moment is

$$M = -a\mathbf{r}_2 \times \left[-mR\left(\ddot{\mathbf{r}}_1 + \frac{g}{R}\mathbf{r}_1\right)\right] \tag{4.4}$$

After calculations

$$M_x = -\frac{amR}{|\dot{\mathbf{r}}_2|}(\mathbf{r}_2 \times \dot{\mathbf{r}}_2)\cdot\left[\ddot{\mathbf{r}}_1 + \frac{g}{R}\mathbf{r}_1\right], \qquad M_y = \frac{amR}{|\dot{\mathbf{r}}_2|}\dot{\mathbf{r}}_2\cdot\left[\ddot{\mathbf{r}}_1 + \frac{g}{R}\mathbf{r}_1\right] \tag{4.5}$$

After substituting (4.2) and (4.5) into (4.3) we have

$$\frac{2B\cos\varepsilon}{amR} = |\dot{\mathbf{r}}_2|, \qquad \omega_x = 0$$
$$\frac{d}{dt}(2B\cos\varepsilon) = \frac{amR}{\dot{\mathbf{r}}_2|}\dot{\mathbf{r}}_2\cdot\left[\ddot{\mathbf{r}}_1 + \frac{g}{R}\mathbf{r}_1\right] \tag{4.6}$$
$$\frac{\ddot{\mathbf{r}}_2\cdot(\mathbf{r}_2\times\dot{\mathbf{r}}_2)}{\dot{\mathbf{r}}_2^2}2B\cos\varepsilon = \frac{amR}{|\dot{\mathbf{r}}_2|}(\mathbf{r}_2\times\dot{\mathbf{r}}_2)\cdot\left[\ddot{\mathbf{r}}_1 + \frac{g}{R}\mathbf{r}_1\right]$$

Eliminating $2B\cos\epsilon/amR$ from the last two equations in system (4.6), simplifying and considering that

$$|\dot{\mathbf{r}}_2|\frac{d|\dot{\mathbf{r}}_2|}{dt} = \frac{1}{2}\frac{d}{dt}\dot{\mathbf{r}}_2^2 = \dot{\mathbf{r}}_2\cdot\ddot{\mathbf{r}}_2$$

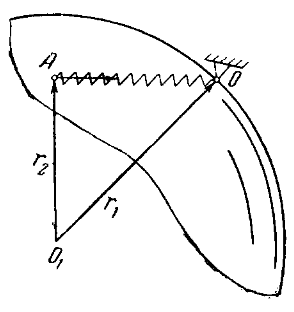

Fig. 5.

we obtain

$$\ddot{\mathbf{r}}_2\cdot\dot{\mathbf{r}}_1 = \left[\ddot{\mathbf{r}}_1 + \frac{g}{R}\mathbf{r}_1\right]\cdot\dot{\mathbf{r}}_1$$

$$\ddot{\mathbf{r}}_2\cdot(\mathbf{r}_1\times\dot{\mathbf{r}}_1) = \left[\ddot{\mathbf{r}}_1 + \frac{g}{R}\mathbf{r}_1\right]\cdot(\mathbf{r}_1\times\dot{\mathbf{r}}_1)$$

This yields the following differential equation:

$$\ddot{\mathbf{r}}_2 + \alpha \mathbf{r}_2 = \ddot{\mathbf{r}}_1 + \frac{g}{R}\mathbf{r}_1, \quad |\mathbf{r}_1| = |\mathbf{r}_2| = 1 \tag{4.7}$$

where a is the constraint reaction $|\mathbf{r}_2| = 1$.

Consider now the following mechanical model (Fig. 5). Point A of a unit mass is located on a unit sphere and is attracted to another point O on the sphere by the spring with rigidity g/R. Point O on the sphere is fixed; O_1 is the sphere center.

Denote unit vectors O_1O and O_1A by ρ_1 and ρ_2, respectively, and the vector OA as ρ. The motion of the sphere is defined by the vector $\rho_1(t)$. The equation of motion for point A is

$$\ddot{\rho} = -\frac{g}{R}\rho - \alpha_1 \rho_2$$

but since $\rho = \rho_2 - \rho_1$

$$\ddot{\rho}_2 + \alpha\rho_2 = \ddot{\rho}_1 + \frac{g}{R}\rho_1 \tag{4.8}$$

Equation (4.8) coincides with Equation (4.7), where ρ_1 is analogous to $\mathbf{r}_1$ and ρ_2 to $\mathbf{r}_2$, and therefore in Fig. 5 $\mathbf{r}_1$ and $\mathbf{r}_2$ are used. Thus, the considered model which represents the Schuler pendulum is a mechanical analog for an ideal horizon gyrocompass.

According to this model, the material point is in a potential field controlled by an ideal but nonstationary constraint. In the particular case of a simple course (rest or motion along a parallel with $\mathbf{r}$ = const) the uniformly rotating vector $\mathbf{r}_1(t)$ describes a cone of rotation (Fig. 6). In a system of coordinates rotating with a definite velocity about the cone axis, the constraint is stationary and the force field remains potential, since to the central field is added only a centrifugal force field also possessing a potential and the Coriolis forces which do no work at all. Consequently there is an energy integral in this case. This integral can be obtained directly as follows. Introduce vector ω, such that

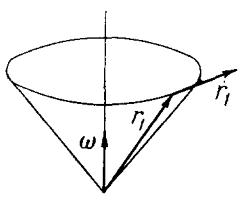

Fig. 6.

$$\mathbf{r}_1 = \boldsymbol{\omega} \times \mathbf{r}_1 \tag{4.9}$$

In the general case $\omega = \mathbf{r}_1 \times \mathbf{r}_1 + \gamma\mathbf{r}_1$ for any $\mathbf{r}_1$ and γ. Perform scalar multiplication of (4.7) by the vector

$$\dot{\mathbf{r}}_2 - \boldsymbol{\omega} \times \mathbf{r}_2$$

Using (4.9) we obtain

$$\frac{d}{dt}\frac{1}{2}\left\{[\dot{\mathbf{r}}_2-\dot{\mathbf{r}}_1+(\mathbf{r}_2-\mathbf{r}_1)\times\boldsymbol{\omega}]^2+\left(\frac{g}{R}-\boldsymbol{\omega}^2\right)(\mathbf{r}_2-\mathbf{r}_1)^2+[\boldsymbol{\omega}\cdot(\mathbf{r}_2-\mathbf{r}_1)]^2\right\}+ \\ +\dot{\boldsymbol{\omega}}\cdot\{(\dot{\mathbf{r}}_2-\dot{\mathbf{r}}_1)\times(\mathbf{r}_2-\mathbf{r}_1)\}=0 \qquad (4.10)$$

The relationship (4.10) is valid for any maneuver $\mathbf{r}_1(t)$ and for any ω satisfying (4.9). For a simple course one may take as ω a vector directed along the stationary cone axis described by $\mathbf{r}_1$; in view of the constancy of rotation, $|\omega|$ = const, i.e. $\dot{\omega} = 0$. Expression (4.10) then becomes the energy integral

$$[\dot{\mathbf{r}}_2-\dot{\mathbf{r}}_1+(\mathbf{r}_2-\mathbf{r}_1)\times\boldsymbol{\omega}]^2+\left(\frac{g}{R}-\boldsymbol{\omega}^2\right)(\mathbf{r}_2-\mathbf{r}_1)^2+[\boldsymbol{\omega}\cdot(\mathbf{r}_2-\mathbf{r}_1)]^2=\text{const} \qquad (4.11)$$

proving the stability of solution $\mathbf{r}_2(t) = \mathbf{r}_1(t)$ for all ω and for

$$|\boldsymbol{\omega}|<\sqrt{g/R}.$$

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